Existence, uniqueness and finite element approximation of the solution of time-harmonic electromagnetic boundary value problems involving metamaterials

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Abstract

Existence and uniqueness of the solution of time-harmonic electromagnetic boundary value problems is analyzed together with the convergence of Galerkin finite element approximations. Sufficient conditions based on the presence of different types of losses and on the properties of the hermitian symmetric parts of the effective dielectric permittivity and the effective magnetic permeability are provided. Metamaterials such as double-negative, epsilon-negative and mu-negative substances are covered by our analysis since any hypothesis on the positive definiteness of the aforementioned hermitian symmetric parts is avoided on purpose.

Index terms - Metamaterials, double-negative materials, epsilon-negative materials, mu-negative materials, existence, uniqueness, finite element approximation, convergence.

1 Introduction

Recently the interest for substances characterized by unusual electromagnetic properties has grown in an impressive way due to the possibility of developing composite materials having response functions that do not occur, or are not readily available, in nature [1]. Such "metamaterials" [1] can be isotropic and have simultaneously negative the real parts of both the permittivity and permeability properties [1]. In this case they are often called double-negative (DNG) [1] materials. Other isotropic metamaterials are known as epsilon-negative (ENG) substances or mu-negative (MNG) materials [2] and are characterized by a negative real part of the effective permittivity and a positive real part of the effective permeability or by a negative real part of the effective permeability and a positive real part of the effective permittivity, respectively [1], [2]. A lot of anisotropic metamaterials are considered in the open literature [3], [4], as well.

Many research groups have proposed interesting applications [1]. In such studies the basic electromagnetic phenomena occurring in these "strange" substances are often analyzed by using quite simplified models, making it possible to calculate the solution of the electromagnetic problem of interest analytically [2] or by truncated series [5].

The emphasis should now increasingly be placed on carrying out more detailed analysis and on finding new, tangible applications that will enable this technology to meet its full potential. In this context, numerical methods should become more and more important [3]. Since the finite element method is one of the most important technique for the numerical solution of electromagnetic problems [6] it seems important to have results on the reliability of this numerical technique when metamaterials are involved.

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Even though many results on the convergence of the finite element approximation of the solution of electromagnetic boundary value problems or electromagnetic eigenproblems are now available [7], [8], [9], [10], [11], [12], [13], all these results, to the best of the authors' knowledge, are obtained by assuming that all materials involved have uniformly positive definite hermitian symmetric parts of the effective dielectric permittivity and effective magnetic permeability, thus preventing their applications to cases where metamaterials play a role.

The target of this work is to overcome this lack of results, at least partially, and prove that many radiation or scattering models [14], [5], and many models of microwave components involving possibly anisotropic metamaterials [15] admit a unique solution which depends continuously on the data (sources of the problems) and can be reliably approximated by the finite element method. In particular, the main result of this paper states that these good features of the models and of the finite element approximations of their solutions hold true *whenever the effective dielectric permittivity is uniformly positive definite on the regions where no losses are modelled in it and, moreover, the effective magnetic permeability is uniformly negative definite on the regions where no losses are modelled in it. The same good features hold true if "positive" is replaced by "negative" and viceversa in the previous sentence.*

Some important things are worth mentioning in order to state more explicitly the scope of this result:

1. The above result gives *sufficient conditions* for existence, uniqueness and finite element approximability of the solutions of the considered boundary value problems.

2. These conditions *characterize all the cases* in which the coerciveness [16] of the sesquilinear form involved in the variational formulation of the problem is ensured and, then, Lax-Milgram and first Strang lemmas can be applied to prove existence, uniqueness and approximability.

3. These conditions are *sufficient* to ensure that resonant modes cannot occur.

4. Problems involving any configuration of different media, some of them being possibly metamaterials, are covered by the above result, provided that the used model takes into account at least a suitably chosen subset of the possible losses.

5. For a given choice of the losses to be taken into account in the model, not all the possible configurations of media are covered by the above result.

When the aforementioned conditions are not satisfied and Lax-Milgram and first Strang lemmas do not apply, the existence, uniqueness and finite element approximability of the solution may yet hold true. However, since resonant modes may take place in this case, more sophisticated mathematical tools, like Fredholm alternative [9], [17] and compactness results [9], [18], [19], are expected to be crucial tools in any attempt to generalize our result. Problem solvability has not been investigated, so far, when metamaterials are involved and resonant modes may take place and the apparently non trivial analysis of this situation surely deserve consideration for future research work.

This paper is organized as follows. In Section 2 the electromagnetic boundary value problem of interest is defined and its variational formulation is provided in Section 3. The well posedness of many realistic models involving metamaterials is discussed in Section 4. The convergence of Galerkin approximations and of finite element approximations is proved in Sections 5 and 6, respectively. Finally, before concluding the paper, some practical implications on the usefulness of our result are pointed out in Section 7. The longest and most technical proof of Section 4 is reported in Appendix A, whereas in Appendix B we provide some considerations on how sharp our theoretical results are.

2 Problem definition

Let Ω be the open, bounded and connected subset of $\mathbb{R}^3$ where the electromagnetic boundary value problem of interest will be posed. Let $\Gamma = \partial\Omega$ be its Lipschitz continuous boundary [20] (p. 4) and denote by $\mathbf{n}$ the outward unit vector normal to Γ. To shorten many statements we summarize the above hypotheses on the domain and its boundary as follows

H1. $\Omega \subset \mathbb{R}^3$ is open, bounded and connected,

H2. $\Gamma = \partial\Omega$ is Lipschitz continuous.

Different inhomogeneous anisotropic materials will be modelled by assuming, without loss of generality, that Ω can be decomposed into m subdomains (open and connected subsets of Ω having Lipschitz continuous boundaries) denoted Ω_i, $i \in M = \{1, \ldots, m\}$, [21] satisfying $\overline{\Omega} = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_m$ ($\overline{\Omega}$ is the closure of Ω) and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. If ε and μ are two 3-by-3 matrix-valued complex functions (with domain $D = \Omega_1 \cup \cdots \cup \Omega_m$) representing the effective [22], [1] dielectric permittivity and the effective magnetic permeability, respectively, we assume moreover that [9] (p. 36)

H3. $\varepsilon|_{\Omega_k} \in (C^0(\overline{\Omega}_k))^{3\times 3}, k \in M$,

H4. $\mu|_{\Omega_k} \in (C^0(\overline{\Omega}_k))^{3\times 3}, k \in M$.

Let us point out that such hypotheses are in no way restrictive for all applications of interest since the material properties are just piecewise but not globally continuous [17].

In order to define the problem of interest we introduce the following additional notations and hypothesis. The symbol ω represents the angular frequency, which, without loss of generality for wave problems, is assumed to satisfy

H5. $\omega \in \mathbb{R},\ \omega > 0$.

Moreover, $\mathbf{J}_e$ and $\mathbf{J}_m$ are the electric and magnetic current densities, respectively, prescribed by the sources, ξ is the scalar complex admittance involved in impedance boundary condition and $\mathbf{f}_R$ is the corresponding inhomogeneous term. Then we look for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ solving the following electromagnetic boundary value problem

$$\begin{cases} \operatorname{curl}\mathbf{H} - j\omega\varepsilon\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \operatorname{curl}\mathbf{E} + j\omega\mu\mathbf{H} = -\mathbf{J}_m & \text{in } \Omega \\ \mathbf{H}\times\mathbf{n} - \xi\,\mathbf{n}\times\mathbf{E}\times\mathbf{n} = \mathbf{f}_R & \text{on } \Gamma. \end{cases} \tag{1}$$

Our assumptions on all terms involved will be made more precise later on.

Note that the third equation in (1) can be used to enforce lowest order absorbing boundary conditions [6] (p. 9), so that the above model can be thought of as an approximation of a radiation problem, or boundary conditions at imperfectly conducting surfaces [23] (pp. 384-385), so that the above model can be thought of as a realistic formulation of a cavity problem.

More complex models could be considered as well. However, we chose the above simple model since, on the one hand, the generality of our results is not reduced in a significant way and, on the other hand, the mathematical developments can be limited to a reasonable extent.

The main objective of this work is to find some simple sufficient conditions under which the above problem is well posed (i.e., it has a unique solution which continuously depends on the data) and its solution can be approximated by a Galerkin finite element method.

3 Variational formulation

The electromagnetic boundary value problem of interest indicated above can be precisely stated by using its variational formulation. In order to write such a formulation it is necessary to introduce some Hilbert spaces [9]. $(L^2(\Omega))^3$ is the usual space of square integrable vector fields on Ω with scalar product given by $(\mathbf{u}, \mathbf{v})_{0,\Omega} = \int_\Omega \mathbf{v}^*\mathbf{u}\; dV$, where $\mathbf{v}^*$ denotes the conjugate transpose of the column vector $\mathbf{v}$. In order to deal with the tangential vector fields involved in the boundary condition we define [9] (p. 48)

$$L_t^2(\Gamma) = \{\mathbf{v} \in (L^2(\Gamma))^3 \mid \mathbf{v} \cdot \mathbf{n} = 0 \ \text{ on } \Gamma\}, \tag{2}$$

with scalar product denoted by $(\mathbf{u}, \mathbf{v})_{0,\Gamma} = \int_\Gamma \mathbf{v}^* \mathbf{u} \; dS$. The space where we should seek the solution is [9] (p. 82; see also p. 69)

$$V = H_{L^2,\Gamma}(\text{curl}, \Omega) = \{\mathbf{v} \in H(\text{curl}, \Omega) \mid \mathbf{v} \times \mathbf{n}_{|\Gamma} \in L_t^2(\Gamma)\}, \tag{3}$$

being $H(\text{curl}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 \mid \text{curl}\, \mathbf{v} \in (L^2(\Omega))^3\}$. The scalar product in the Hilbert space V is given by [9] (p. 84, p. 69)

$$(\mathbf{u}, \mathbf{v})_{V,\Omega} = (\mathbf{u}, \mathbf{v})_{0,\Omega} + (\text{curl}\, \mathbf{u}, \text{curl}\, \mathbf{v})_{0,\Omega} + (\mathbf{n} \times \mathbf{u}, \mathbf{n} \times \mathbf{v})_{0,\Gamma}, \tag{4}$$

and the induced norm is $\|\mathbf{u}\|_{V,\Omega} = (\mathbf{u}, \mathbf{u})_{V,\Omega}^{1/2}$.

Finally, the admittance function ξ with domain Γ and range in $\mathbb{C}$ is assumed to satisfy

H6. ξ is piecewise continuous and bounded.

This hypothesis is satisfied when lowest order absorbing boundary conditions or boundary conditions at imperfectly conducting surfaces are considered, as can be easily verified.

Under all the above hypotheses, by defining the sesquilinear form

$$a(\mathbf{u}, \mathbf{v}) = (\mu^{-1} \text{curl}\, \mathbf{u}, \text{curl}\, \mathbf{v})_{0,\Omega} - \omega^2 (\varepsilon \mathbf{u}, \mathbf{v})_{0,\Omega} + j\omega(\xi\, \mathbf{n} \times \mathbf{u} \times \mathbf{n}, \mathbf{n} \times \mathbf{v} \times \mathbf{n})_{0,\Gamma} \quad \mathbf{u}, \mathbf{v} \in V \tag{5}$$

and, for any given $\mathbf{J}_e \in (L^2(\Omega))^3$, $\mathbf{J}_m \in (L^2(\Omega))^3$ and $\mathbf{f}_R \in L_t^2(\Gamma)$ the antilinear form

$$l(\mathbf{v}) = -j\omega(\mathbf{J}_e, \mathbf{v})_{0,\Omega} - (\mu^{-1} \mathbf{J}_m, \text{curl}\, \mathbf{v})_{0,\Omega} - j\omega(\mathbf{f}_R, \mathbf{n} \times \mathbf{v} \times \mathbf{n})_{0,\Gamma} \quad \mathbf{v} \in V, \tag{6}$$

the variational formulation of the problem of interest is [9] (pp. 81-82)

Problem 1. *Given $\omega \in \mathbb{R}, \omega > 0$, $\mathbf{J}_e \in (L^2(\Omega))^3$, $\mathbf{J}_m \in (L^2(\Omega))^3$, $\mathbf{f}_R \in L_t^2(\Gamma)$, find $\mathbf{E} \in V$ such that*

$$a(\mathbf{E}, \mathbf{v}) = l(\mathbf{v}) \quad \forall \mathbf{v} \in V \tag{7}$$

where our hypotheses are sufficient to give a meaning to all terms except for the ones involving μ^{-1}. Some additional hypotheses which, as a by-product, guarantee the invertibility of μ will be considered in the next section.

4 Further hypotheses for well posedness

In order to give a meaning to all terms involving μ^{-1} in Problem 1 it would be possible to directly assume the invertibility of μ. However, this simple hypothesis is not sufficient to obtain a result on the well posedness of the problem itself. For this reason in the following we analyse the effects of stronger hypotheses.

As already pointed out the sharper technique based on the Fredholm alternative [9] (p. 83 and p. 24) [17] will not be considered here to keep the mathematical complexity to a minimum. We will consider, instead, the technique based on the Lax-Milgram theorem [16], [24] (pp. 376-377). It requires the continuity and the coercivity of the sesquilinear form on the left hand side of equation (7) and the continuity of the antilinear form on the right hand side of the same equation. From this point of view, in some way, this paper is a simple analysis of the conditions which guarantee that the sesquilinear form on the left hand side of equation (7) is coercive, being the two continuity conditions almost trivial.

In order to obtain such a result, we firstly state a theorem which would be obvious in the case of isotropic media. It is provided without proof but similar results can be found in [25] (pp. 21-22). To state the theorem we need some more definitions. We will say that a 3×3 hermitian symmetric matrix ζ is positive (negative) semidefinite if $\mathbf{v}^*\zeta\mathbf{v} \geq 0$ ($\mathbf{v}^*\zeta\mathbf{v} \leq 0$), $\forall \mathbf{v} \in \mathbb{C}^3$. A 3×3 hermitian symmetric matrix-valued function $\zeta(\mathbf{x})$ with domain D, is said to be uniformly positive (negative) definite on an open set $D_o \subset D$ if there exists $C > 0$ such that $\mathbf{v}^*\zeta(\mathbf{x})\mathbf{v} \geq C|\mathbf{v}|^2$ ($\mathbf{v}^*\zeta(\mathbf{x})\mathbf{v} \leq -C|\mathbf{v}|^2$) $\forall \mathbf{x} \in D_o$, $\forall \mathbf{v} \in \mathbb{C}^3$.

We have

Theorem 1. *Let $D_o \subseteq \Omega_i$, $1 \leq i \leq m$, D_o open.*

The matrix-valued function μ has an inverse $\nu = \mu^{-1}$ on D_o whenever at least one of the hermitian symmetric matrix-valued functions $\zeta_1 = \frac{\mu+\mu^*}{2}$ or $\zeta_2 = \frac{\mu^*-\mu}{2j}$ is either uniformly positive or uniformly negative definite on D_o. If, moreover, $\mu|_{D_o} \in (C^0(\overline{D}_o))^{3\times 3}$ then $\nu|_{D_o} \in (C^0(\overline{D}_o))^{3\times 3}$.

The hermitian symmetric matrix-valued function $\zeta_3 = \frac{\nu+\nu^*}{2}$ is uniformly positive (negative) definite on D_o if ζ_1 is uniformly positive (negative) definite on D_o and the entries of μ are bounded on D_o. On the contrary, the hermitian symmetric matrix-valued function $\zeta_4 = \frac{\nu^*-\nu}{2j}$ is uniformly negative (positive) definite on D_o if ζ_2 is uniformly positive (negative) definite on D_o and the entries of μ are bounded on D_o. Furthermore, ζ_4 is negative (positive) semidefinite on D_o if ν exists and ζ_2 is positive (negative) semidefinite on D_o.

Since all materials we are interested in are passive we immediately have that the matrix-valued functions $\frac{\varepsilon^*-\varepsilon}{2j}$ and $\frac{\mu^*-\mu}{2j}$ satisfy the following condition [26] (pp. 48-49)

H7. $\mathbf{v}^* \frac{\varepsilon^*(\mathbf{x})-\varepsilon(\mathbf{x})}{2j}\mathbf{v} \geq 0$ and $\mathbf{v}^* \frac{\mu^*(\mathbf{x})-\mu(\mathbf{x})}{2j}\mathbf{v} \geq 0$, $\forall \mathbf{v} \in \mathbb{C}^3$ and $\forall \mathbf{x} \in \Omega_i$, $i \in M$.

However, in all regions where the above hermitian forms are equal to zero the substances are lossless [26] (pp. 48-49).

We firstly consider the following condition on μ

H8. For any given i, $i \in \mathbb{N}$, $1 \leq i \leq m$, $\frac{\mu^*-\mu}{2j}$ is uniformly positive definite in Ω_i or $\frac{\mu+\mu^*}{2}$ is uniformly positive definite in Ω_i or $\frac{\mu+\mu^*}{2}$ is uniformly negative definite in Ω_i.

We have

Theorem 2. *Whenever H1, H2, H3, H4, H6 and H8 are satisfied, the sequilinear form $a(\mathbf{u}, \mathbf{v})$ is continuous on $V \times V$ and the antilinear form $l(\mathbf{v})$ is continuous on V if $\mathbf{J}_e \in (L^2(\Omega))^3$, $\mathbf{J}_m \in (L^2(\Omega))^3$ and $\mathbf{f}_R \in L^2_t(\Gamma)$.*

Proof. The hypotheses H1 and H2 are necessary to give a precise meaning to the space V [9]. By using H4, H8 and Theorem 1 we deduce that μ^{-1} exists and that $\mu^{-1}|_{\Omega_i} \in (C^0(\overline{\Omega}_i))^{3\times 3}$. Then $\mu^{-1}\mathbf{u} \in (L^2(\Omega))^3$ and $\exists C > 0 : \|\mu^{-1}\mathbf{u}\|_{0,\Omega} \leq C\|\mathbf{u}\|_{0,\Omega}$ for any $\mathbf{u} \in (L^2(\Omega))^3$. Hypotheses H3 and H6 are used to obtain analogous result on $\varepsilon\mathbf{u}$ and $\xi\, \mathbf{n} \times \mathbf{u} \times \mathbf{n}$, respectively. Both continuities are now a consequence of the Cauchy-Schwartz inequality. $\square$

In order to prove that $a(\mathbf{u}, \mathbf{v})$ is coercive on V [24] (p. 369) (i.e., $\exists C > 0$, such that $|a(\mathbf{u}, \mathbf{u})| \geq C\|\mathbf{u}\|^2_{V,\Omega}$ $\forall \mathbf{u} \in V$) further properties of ε and μ are necessary, as it will be proved later on. With this aim, let us define $D_{ml} = \cup_{i\in J_{ml}}\Omega_i$, $J_{ml} \subset M$, as the union of the subdomains Ω_i where $\frac{\mu^*-\mu}{2j}$ is uniformly positive definite (there are losses hidden in μ), $D_{mp} = \cup_{i\in J_{mp}}\Omega_i$, $J_{mp} \subset M \setminus J_{ml}$, as the union of the subdomains $\Omega_i \not\subset D_{ml}$ where $\frac{\mu+\mu^*}{2}$ is uniformly positive definite, and $D_{mn} = \cup_{i\in J_{mn}}\Omega_i$, $J_{mn} \subset M \setminus J_{ml}$, as the union of the subdomains $\Omega_i \not\subset D_{ml}$ where $\frac{\mu+\mu^*}{2}$ is uniformly negative definite. Let, moreover, $D_{el} = \cup_{i\in J_{el}}\Omega_i$, $J_{el} \subset M$ be the union of the subdomains Ω_i where $\frac{\varepsilon^*-\varepsilon}{2j}$ is uniformly positive definite (there are losses hidden in ε), $D_{ep} = \cup_{i\in J_{ep}}\Omega_i$, $J_{ep} \subset M \setminus J_{el}$ be the union of the subdomains $\Omega_i \not\subset D_{el}$ where $\frac{\varepsilon+\varepsilon^*}{2}$ is uniformly positive definite, and $D_{en} = \cup_{i\in J_{en}}\Omega_i$, $J_{en} \subset M \setminus J_{el}$ be the union of the subdomains $\Omega_i \not\subset D_{el}$ where $\frac{\varepsilon+\varepsilon^*}{2}$ is uniformly negative definite. Assume that

H9. $\big(J_{mp} = M \setminus J_{ml}$ and $J_{en} = M \setminus J_{el}\big)$ or $\big(J_{mn} = M \setminus J_{ml}$ and $J_{ep} = M \setminus J_{el}\big)$.

Remark 1. *Condition H8 can be written also as* $J_{ml} \cup J_{mp} \cup J_{mn} = M$ *and holds true whenever condition H9 is satisfied.*

A condition on the admittance function ξ could be of a type similar to H9 and our technique would work also in that case. However, we decided to consider just the following simpler assumption of a "everywhere lossy boundary"

H10. $\exists \alpha \in \mathbb{R}$, $\alpha > 0$, such that $Re(\xi(\mathbf{x})) \geq \alpha$ for all $\mathbf{x} \in \Gamma$,

since it is sufficient to cover both the lowest order absorbing boundary condition and boundary conditions at imperfectly conducting surfaces, while, if allowing the most general case, the analytical details would become heavier, but just a few practically significant cases involving perfect electric or magnetic conductors would be additionally covered.

Now we are in a position to state the following theorem, which is proved in Appendix A.

Theorem 3. *Whenever H1, H2, H3, H4, H5, H6, H7 and H10 are satisfied, the sequilinear form* $a(\mathbf{u}, \mathbf{v})$ *is coercive on* V *if condition H9 holds true.*

To conclude this section, we collect in a single statement the main result of this section

Theorem 4. *Whenever H1, H2, H3, H4, H5, H6, H7, H9 and H10 are satisfied, Problem 1 is well posed.*

Proof. This is a consequence of Remark 1, Theorems 2 and 3 and the Lax-Milgram theorem. □

5 Galerkin approximation

In engineering practice a widely used approach for approximating Problem 1 is Galerkin's method [27] (p. 59), which consists in subtituting Problem 1 by a similar problem posed in a finite dimensional subspace V_h of V, which we refer to as the discrete problem.

Convergence [27] (p. 112), which is the main feature an approximation must have to be considered satisfactory, is then defined as a property of the sequence of solutions of the discrete problems generated by a sequence of finite dimensional subspaces of V denoted by $\{V_h\}$, $h \in I$, where I is a denumerable and bounded set of strictly positive indexes having zero as the only limit point [27] (p. 112).

For any $h \in I$ and for any set of approximate sources $\mathbf{J}_{eh} \in (L^2(\Omega))^3$, $\mathbf{J}_{mh} \in (L^2(\Omega))^3$ and $\mathbf{f}_{Rh} \in L^2(\Gamma)^3$ we define the antilinear form

$$l_h(\mathbf{v}_h) = -j\omega(\mathbf{J}_{eh}, \mathbf{v}_h)_{0,\Omega} - (\mu^{-1}\mathbf{J}_{mh}, \operatorname{curl} \mathbf{v}_h)_{0,\Omega} - j\omega(\mathbf{f}_{Rh}, \mathbf{n} \times \mathbf{v}_h \times \mathbf{n})_{0,\Gamma} \;\; \mathbf{v}_h \in V_h. \tag{8}$$

Then, the aforementioned discrete version of Problem 1 reads as follows

Problem 2. *Given* $\omega \in \mathbb{R}$, $\omega > 0$, $\mathbf{J}_{eh} \in (L^2(\Omega))^3$, $\mathbf{J}_{mh} \in (L^2(\Omega))^3$, $\mathbf{f}_{Rh} \in L^2(\Gamma)^3$, *find* $\mathbf{E}_h \in V_h$ *such that*

$$a(\mathbf{E}_h, \mathbf{v}_h) = l_h(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h. \tag{9}$$

Whenever Problem 1 is well posed, we say that Problem 2 is a convergent approximation of Problem 1 if the sequence $\{\mathbf{E}_h\}$ of solutions of Problem 2 satisfies $\lim_{h\to 0} \|\mathbf{E} - \mathbf{E}_h\|_{V,\Omega} = 0$, $\mathbf{E}$ being the solution of Problem 1.

Let us consider the following hypotheses concerning the approximate sources

H11. $\lim_{h\to 0} \|\mathbf{J}_e - \mathbf{J}_{eh}\|_{0,\Omega} = 0$,

H12. $\lim_{h\to 0} \|\mathbf{J}_m - \mathbf{J}_{mh}\|_{0,\Omega} = 0$,

H13. $\lim_{h\to 0} \|\mathbf{f}_R - \mathbf{f}_{Rh}\|_{0,\Gamma} = 0$.

We suppose, moreover, that the capability of V_h to approximate V becomes more and more satisfactory as h is reduced, in the precise sense given by the following condition (which is indicated by (CAS) in [11])

H14. $\lim_{h\to 0} \inf_{\mathbf{v}_h \in V_h} \|\mathbf{v} - \mathbf{v}_h\|_{V,\Omega} = 0 \quad \forall \mathbf{v} \in V$.

Then we have

Theorem 5. *Whenever H1, H2, H3, H4, H5, H6, H7, H9, H10, H11, H12, H13 and H14 are satisfied, Problem 2 is a convergent approximation of Problem 1.*

Proof. Theorem 4 implies that Problem 1 admits a unique solution $\mathbf{E} \in V$. By using H9, Remark 1, H4 and Theorem 1 we deduce that μ^{-1} exists and that $\mu^{-1}|_{\Omega_i} \in (C^0(\overline{\Omega}_i))^{3\times 3}$. Thus the approximate antilinear form $l_h(\mathbf{v}_h)$ is well defined.

By using the Chauchy-Schwartz inequality we deduce that $\exists C > 0$, such that

$$\begin{aligned} &|l(\mathbf{v}_h) - l_h(\mathbf{v}_h)| = \\ &\quad |-j\omega(\mathbf{J}_e - \mathbf{J}_{eh}, \mathbf{v}_h)_{0,\Omega} - (\mu^{-1}(\mathbf{J}_m - \mathbf{J}_{mh}), \operatorname{curl} \mathbf{v}_h)_{0,\Omega} - j\omega(\mathbf{f}_R - \mathbf{f}_{Rh}, \mathbf{n} \times \mathbf{v}_h \times \mathbf{n})_{0,\Gamma}| \\ &\quad \le C\big(\|\mathbf{J}_e - \mathbf{J}_{eh}\|_{0,\Omega} + \|\mathbf{J}_m - \mathbf{J}_{mh}\|_{0,\Omega} + \|\mathbf{f}_R - \mathbf{f}_{Rh}\|_{0,\Gamma}\big)\|\mathbf{v}_h\|_{V,\Omega} \ \forall \mathbf{v}_h \in V_h. \end{aligned} \tag{10}$$

By Theorem 3 the sesquilinear form appearing on the left hand side of Problems 1 and 2 is coercive on V and then also uniformly coercive on V_h [27] (p. 192). As the remaining assumptions needed to apply the first Strang Lemma [27] (p. 192) are satisfied owing to Remark 1 and Theorem 2, the convergence is a direct consequence of the first Strang Lemma itself, inequality (10) and hypotheses H11, H12, H13 and H14. □

6 Finite element approximation

Very often in practice the sequence of finite dimensional subspaces considered in Galerkin method is built by using the finite element method [27]. This is done, as usual, by considering a sequence of triangulations $\{\mathcal{T}_h\}$, $h \in I$ of $\overline{\Omega}$ [27] (p. 59) and a specific finite element on each triangulation $\mathcal{T}_h$ [27].

In order to avoid the many technicalities involved when curved boundaries are considered [27] (Chapter VI) we assume that [27] (p. 65)

H15. Ω is a polyhedron (i.e., $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} T$).

For finite element approximations it is usual to make the following basic assumption [27] (p. 131) meaning that no element of the triangulation degenerates as $h \to 0$

H16. the family $\{\mathcal{T}_h\}$ of triangulations is regular.

Finally, the electromagnetic community is well aware that Nedelec's edge elements defined on tetrahedra [28] are often the best choice, so that we assume

H17. $\mathcal{T}_h$ is made up of tetrahedra $\forall h \in I$,

H18. Nedelec's edge elements of a given order defined on tetrahedra [28] are used to build V_h, $\forall h \in I$.

Now by using for example Theorem 5.41, Lemma 5.52 and Theorem 3.54 of [9] we classically conclude that

Theorem 6. *Whenever H1, H2, H15, H16, H17 and H18 are satisfied, the space sequence $\{V_h\}$ satisfies condition H14.*

so that, by using Theorem 5, we draw the following conclusion for the finite element approximations considered.

Theorem 7. *Whenever H1, H2, H3, H4, H5, H6, H7, H9, H10, H11, H12, H13, H15, H16, H17 and H18 are satisfied, Problem 2 is a convergent approximation of Problem 1.*

7 Practical implications

Looking at the crucial condition H9, it is apparent that our results apply to any configuration of different media, some of them being possibly metamaterials, provided that at least a suitably chosen minimal subset of the possible losses is taken into account. Any additional losses besides this minimal set can be safely added. Moreover, since magnetic losses are quite unrealistic for the air and many dielectrics, it is a happy accident that in order to satisfy H9 we are never compelled to introduce any magnetic losses, except possibly in metamaterials (in fact, for instance, electric losses everywhere plus magnetic losses where $\frac{\mu+\mu^*}{2}$ is uniformly negative definite always works). Hence, any realistic model can be dealt with. On the contrary, the most challanging models are those involving ideal lossless materials.

In this section we present a couple of practical applications of the theory developed so far, indicating the minimal subsets of the possible losses that must be taken into account.

The first example of interest concerns radiation or scattering problems (see for example [14]). A simplified model is shown in Figure 1 but note that our hypotheses allow the treatment of multiple and/or composite objects of very irregular shapes.

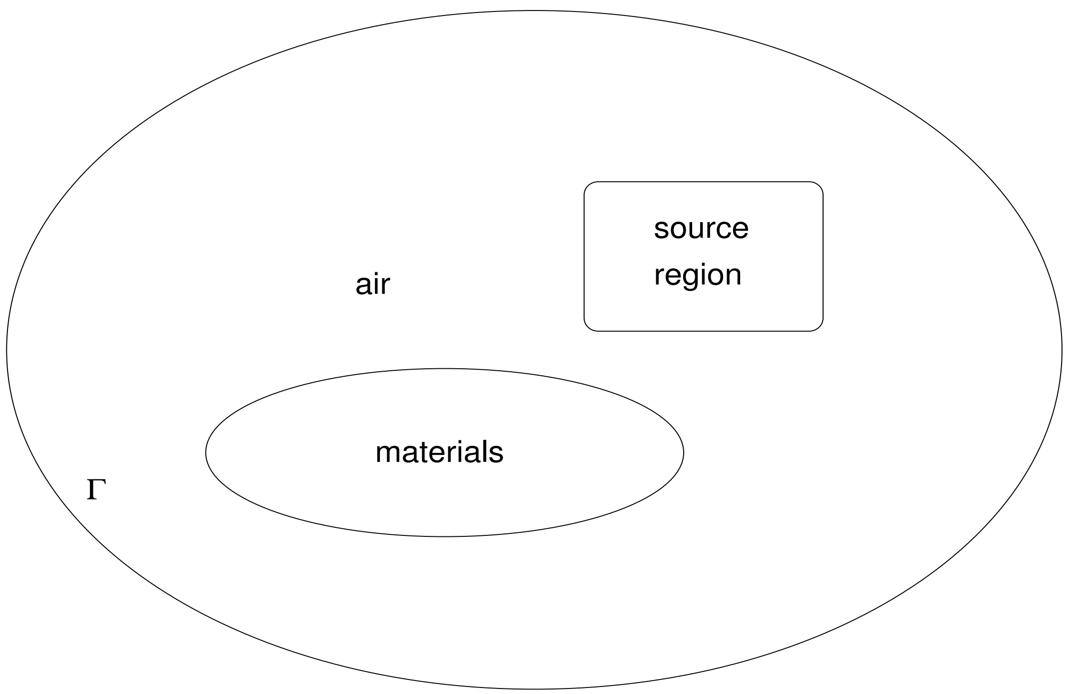

Figure 1: A simple radiation or scattering problem possibly involving metamaterials.

Let us firstly consider the case where only standard materials are involved. By standard materials we mean media having both the hermitian symmetric parts of ε and μ uniformly positive definite in the whole problem domain. Such a case was considered for example by [16] and [8]. However, from Theorem 4 (see also Theorem 11, Appendix B) we can deduce the following stronger conclusion. As a matter of fact, in this case our results apply if

1. in the whole problem domain some losses (however small but strictly positive on the whole region) are taken into account in ε, or

2. in the whole problem domain some losses (however small but strictly positive on the whole region) are taken into account in μ.

Let us now consider problems involving metamaterials. First of all, suppose that the material region of Figure 1 is filled with double-negative materials (where, by extending the usual terminology, we mean materials having the hermitian symmetric parts of ε and μ uniformly negative definite in such region). In such case, from H9 we deduce that our results apply provided that

1. in the air region some losses (however small but strictly positive on the whole region) are taken into account in ε and in the metamaterial region some losses (however small but strictly positive on the whole region) are taken into account in μ, or

2. in the air region some losses (however small but strictly positive on the whole region) are taken into account in μ and in the metamaterial region some losses (however small but strictly positive on the whole region) are taken into account in ε,

being the first possibility of more practical interest than the second one.

Secondly, suppose that the material region is filled with epsilon-negative materials (where, by extending the usual terminology, we mean materials having, in the considered region, the hermitian symmetric part of ε uniformly negative definite and the hermitian symmetric part of μ uniformly positive definite). In such case, from H9 we deduce that our results apply provided that

1. in the air region some losses (however small but strictly positive on the whole region) are taken into account in ε, or

2. in the air region some losses (however small but strictly positive on the whole region) are taken into account in μ and in the metamaterial region some losses (however small but strictly positive on the whole region) are taken into account in both ε and μ.

Finally, suppose that the material region is filled with mu-negative materials (where, by extending the usual terminology, we mean materials having, in the considered region, the hermitian symmetric part of μ uniformly negative definite and the hermitian symmetric part of ε uniformly positive definite). In such case, from H9 we deduce that our results apply provided that

1. in the air region some losses (however small but strictly positive on the whole region) are taken into account in μ, or

2. in the air region some losses (however small but strictly positive on the whole region) are taken into account in ε and in the metamaterial region some losses (however small but strictly positive on the whole region) are taken into account in both ε and μ.

Which losses must be taken into account in more complicate situations (e. g. simultaneously involving metamaterials of different types) in order that our results apply, can be again deduced from H9.

The second example of interest concerns microwave components possibly involving metamaterials. A simplified reference model is shown in Figure 2, where a top view of a rectangular waveguide is considered. By using the formulation provided in [6] (p. 264) as far as the two ports are concerned and considering the walls made up of real conductors the problem can be formulated with equations (1). Exactly the same conclusions as before and the same comments about more complex material configurations do apply. Moreover, they are not affected by the possible addition of any number of ports.

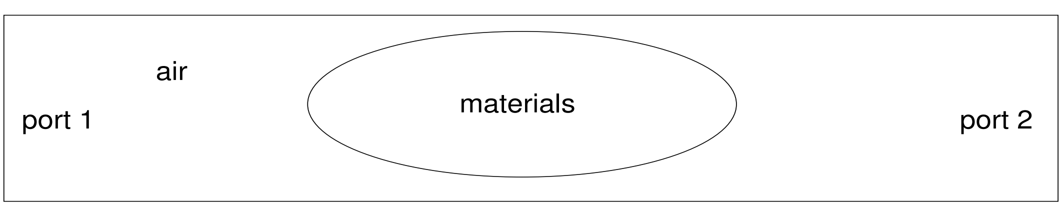

Figure 2: A simple microwave component possibly involving metamaterials.

The reader should be aware that to keep the mathematical complexity to a minimum some ideal models of interest, such as those involving PEC or PMC boundaries, have been excluded.

8 Conclusions

In this paper sufficient conditions for the well-posedness of time-harmonic electromagnetic boundary value problems were provided together with some results on the convergence of their finite element approximations. The presented conditions are not necessary and one should not consider scenarios outside the scope of Theorems 4, 5 or 7 as completely unamenable to finite element simulations. Several practical situations fit the aforementioned theorems.

9 Appendix A

In this Section we provide the proof of Theorem 3.

Proof of Theorem 3. As already pointed out in the proof of Theorem 2 the hypotheses H1 and H2 are considered to give a precise meaning to the space V.

By using H9, Remark 1 and Theorem 1 we deduce that μ^{-1} exists. Let us denote then by ζ_3, ζ_4, ζ_5 and ζ_6 the hermitian symmetric matrix-valued functions $\frac{\mu^{-1}+(\mu^{-1})^*}{2}$, $\frac{(\mu^{-1})^*-\mu^{-1}}{2j}$, $\frac{\varepsilon+\varepsilon^*}{2}$ and $\frac{\varepsilon^*-\varepsilon}{2j}$, respectively. We have

$$|a(\mathbf{u},\mathbf{u})|^2 = |(\mu^{-1}\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,\Omega} - \omega^2(\varepsilon\mathbf{u},\mathbf{u})_{0,\Omega} + j\omega(\xi\,\mathbf{n}\times\mathbf{u}\times\mathbf{n},\mathbf{n}\times\mathbf{u}\times\mathbf{n})_{0,\Gamma}|^2 =$$
$$\Bigg((\zeta_3\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,\Omega} - \omega^2(\zeta_5\mathbf{u},\mathbf{u})_{0,\Omega} - \omega\ Im\Big(\big(\xi\,\mathbf{n}\times\mathbf{u}\times\mathbf{n},\mathbf{n}\times\mathbf{u}\times\mathbf{n}\big)_{0,\Gamma}\Big)\Bigg)^2 +$$
$$\Bigg(-(\zeta_4\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,\Omega} + \omega^2(\zeta_6\mathbf{u},\mathbf{u})_{0,\Omega} + \omega\ Re\Big(\big(\xi\,\mathbf{n}\times\mathbf{u}\times\mathbf{n},\mathbf{n}\times\mathbf{u}\times\mathbf{n}\big)_{0,\Gamma}\Big)\Bigg)^2. \quad (11)$$

By using H10 we obtain $Re\Big(\big(\xi\,\mathbf{n}\times\mathbf{u}\times\mathbf{n},\mathbf{n}\times\mathbf{u}\times\mathbf{n}\big)_{0,\Gamma}\Big) \geq C_2\|\mathbf{n}\times\mathbf{u}\times\mathbf{n}\|^2_{0,\Gamma},\ C_2>0$. Since all materials considered are passive (by H7), by using H9 we have $(\zeta_6\mathbf{u},\mathbf{u})_{0,\Omega} = (\zeta_6\mathbf{u},\mathbf{u})_{0,D_{el}} + (\zeta_6\mathbf{u},\mathbf{u})_{0,D\setminus D_{el}} \geq C_1\|\mathbf{u}\|^2_{0,D_{el}},\ C_1>0$. Moreover, by using again H7, H9 and Theorem 1 we deduce $-(\zeta_4\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,\Omega} = -(\zeta_4\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,D_{ml}} - (\zeta_4\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,D\setminus D_{ml}} \geq C_3\|\mathrm{curl}\,\mathbf{u}\|^2_{0,D_{ml}},\ C_3>0$. Thus, by using also H5 $\exists C_4>0$ such that

$$|a(\mathbf{u},\mathbf{u})|^2 \geq$$
$$\Bigg((\zeta_3\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,\Omega} - \omega^2(\zeta_5\mathbf{u},\mathbf{u})_{0,\Omega} - \omega\ Im\Big(\big(\xi\,\mathbf{n}\times\mathbf{u}\times\mathbf{n},\mathbf{n}\times\mathbf{u}\times\mathbf{n}\big)_{0,\Gamma}\Big)\Bigg)^2 +$$
$$C_4^2\Big(\|\mathrm{curl}\,\mathbf{u}\|^2_{0,D_{ml}} + \|\mathbf{u}\|^2_{0,D_{el}} + \|\mathbf{n}\times\mathbf{u}\times\mathbf{n}\|^2_{0,\Gamma}\Big)^2. \quad (12)$$

The three addends appearing in the first parenthesis on the right hand side are managed as follows. To simplify the notation let $s_c = \omega\ Im\Big(\big(\xi\,\mathbf{n}\times\mathbf{u}\times\mathbf{n},\mathbf{n}\times\mathbf{u}\times\mathbf{n}\big)_{0,\Gamma}\Big)$ and split the first two addends $(\zeta_3\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,\Omega}$ and $\omega^2(\zeta_5\mathbf{u},\mathbf{u})_{0,\Omega}$ as $(\zeta_3\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,D_{ml}} + (\zeta_3\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,D\setminus D_{ml}}$, and $\omega^2(\zeta_5\mathbf{u},\mathbf{u})_{0,D_{el}} + \omega^2(\zeta_5\mathbf{u},\mathbf{u})_{0,D\setminus D_{el}}$, respectively. To shorten the notation, let s_{a1}, s_{a2}, s_{b1} and s_{b2} be equal to $(\zeta_3\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,D_{ml}}$, $(\zeta_3\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{u})_{0,D\setminus D_{ml}}$, $\omega^2(\zeta_5\mathbf{u},\mathbf{u})_{0,D_{el}}$ and $\omega^2(\zeta_5\mathbf{u},\mathbf{u})_{0,D\setminus D_{el}}$, respectively. Define, moreover, $s_b = -s_{a1} + s_{b1} + s_c$ and $s_a = s_{a2} - s_{b2}$ so that the first big parenthesis on the right hand side of inequality (12) is equal to $(s_a - s_b)^2,\ s_a, s_b \in \mathbb{R}$. Since $2st \leq \alpha s^2 + (1/\alpha)t^2,\ \forall\alpha\in\ \mathbb{R},\ \alpha>0$ we obtain

$$(s_a - s_b)^2 = s_a^2 - 2s_as_b + s_b^2 \geq (1-\alpha)s_a^2 + (1-(1/\alpha))s_b^2,\ \forall\alpha\in\ \mathbb{R},\ \alpha>0. \quad (13)$$

In assumption H9 two possibilities are considered. As a matter of fact, by using also Theorem 1, either ζ_3 is uniformly positive definite on $D\setminus D_{ml}$ and ζ_5 is uniformly negative definite on $D\setminus D_{el}$

or ζ_3 is uniformly negative definite on $D \setminus D_{ml}$ and ζ_5 is uniformly positive definite on $D \setminus D_{el}$. In both cases, $\exists C_5 > 0$:

$$|s_a| = |(\zeta_3 \text{curl}\, \mathbf{u}, \text{curl}\, \mathbf{u})_{0,D\setminus D_{ml}} - \omega^2(\zeta_5 \mathbf{u}, \mathbf{u})_{0,D\setminus D_{el}}| \geq C_5\big(\|\text{curl}\, \mathbf{u}\|^2_{0,D\setminus D_{ml}} + \|\mathbf{u}\|^2_{0,D\setminus D_{el}}\big). \quad (14)$$

Moreover, by using H3 $\exists C_6 > 0$ such that

$$|s_{b1}| = \omega^2 \big|(\zeta_5 \mathbf{u}, \mathbf{u})_{0,D_{el}}\big| \leq C_6 \|\mathbf{u}\|^2_{0,D_{el}} \quad (15)$$

and by H6 $\exists C_7 > 0$ such that

$$|s_c| = \omega \left| Im\Big(\big(\xi\, \mathbf{n} \times \mathbf{u} \times \mathbf{n}, \mathbf{n} \times \mathbf{u} \times \mathbf{n}\big)_{0,\Gamma}\Big)\right| \leq C_7 \|\mathbf{n} \times \mathbf{u} \times \mathbf{n}\|^2_{0,\Gamma}. \quad (16)$$

Finally, by H9, H4 and Theorem 1 we have that $\zeta_3 \in (C^0(\overline{D}_{ml}))^{3\times 3}$. Thus, $\exists C_8 > 0$ such that

$$|s_{a1}| = |(\zeta_3 \text{curl}\, \mathbf{u}, \text{curl}\, \mathbf{u})_{0,D_{ml}}| \leq C_8 \|\text{curl}\, \mathbf{u}\|^2_{0,D_{ml}}. \quad (17)$$

Thus, by inequalities (15), (16) and (17), $\exists C_9 > 0$ such that

$$|s_b| = |-s_{a1} + s_{b1} + s_c| \leq C_9 \Big(\|\text{curl}\, \mathbf{u}\|^2_{0,D_{ml}} + \|\mathbf{u}\|^2_{0,D_{el}} + \|\mathbf{n} \times \mathbf{u} \times \mathbf{n}\|^2_{0,\Gamma}\Big). \quad (18)$$

For all $\alpha \in \mathbb{R},\ 0 < \alpha < 1$, we have $(1-\alpha) > 0$ and $(1-(1/\alpha)) < 0$. Then, by inequalities (12), (13), (14) and (18) we deduce $\forall \alpha \in \mathbb{R},\ 0 < \alpha < 1$,

$$\begin{aligned}
|a(\mathbf{u},\mathbf{u})|^2 \geq\ & (s_a - s_b)^2 + C_4^2 \Big(\|\text{curl}\, \mathbf{u}\|^2_{0,D_l} + \|\mathbf{u}\|^2_{0,\Omega} + \|\mathbf{n} \times \mathbf{u} \times \mathbf{n}\|^2_{0,\Gamma}\Big)^2 \geq \\
& (1-\alpha) s_a^2 + (1-(1/\alpha)) s_b^2 + C_4^2 \Big(\|\text{curl}\, \mathbf{u}\|^2_{0,D_{ml}} + \|\mathbf{u}\|^2_{0,D_{el}} + \|\mathbf{n} \times \mathbf{u} \times \mathbf{n}\|^2_{0,\Gamma}\Big)^2 \geq \\
& (1-\alpha) C_5^2 \big(\|\text{curl}\, \mathbf{u}\|^2_{0,D\setminus D_{ml}} + \|\mathbf{u}\|^2_{0,D\setminus D_{el}}\big)^2 + \\
& (1-(1/\alpha)) C_9^2 \Big(\|\text{curl}\, \mathbf{u}\|^2_{0,D_{ml}} + \|\mathbf{u}\|^2_{0,D_{el}} + \|\mathbf{n} \times \mathbf{u} \times \mathbf{n}\|^2_{0,\Gamma}\Big)^2 + \\
& C_4^2 \Big(\|\text{curl}\, \mathbf{u}\|^2_{0,D_{ml}} + \|\mathbf{u}\|^2_{0,D_{el}} + \|\mathbf{n} \times \mathbf{u} \times \mathbf{n}\|^2_{0,\Gamma}\Big)^2 = \qquad (19) \\
& (1-\alpha) C_5^2 \big(\|\text{curl}\, \mathbf{u}\|^2_{0,D\setminus D_{ml}} + \|\mathbf{u}\|^2_{0,D\setminus D_{el}}\big)^2 + \qquad (20) \\
& \Big(C_4^2 + (1 - \frac{1}{\alpha}) C_9^2\Big) \Big(\|\text{curl}\, \mathbf{u}\|^2_{0,D_{ml}} + \|\mathbf{u}\|^2_{0,D_{el}} + \|\mathbf{n} \times \mathbf{u} \times \mathbf{n}\|^2_{0,\Gamma}\Big)^2.
\end{aligned}$$

By choosing $1 > \alpha > \frac{C_9^2}{C_4^2 + C_9^2} > 0$ both coefficients of the last member are positive so that $\exists C_{10} > 0$ such that

$$\begin{aligned}
&|a(\mathbf{u},\mathbf{u})|^2 \geq \qquad (21) \\
&\quad C_{10} \left(\big(\|\text{curl}\, \mathbf{u}\|^2_{0,D\setminus D_{ml}} + \|\mathbf{u}\|^2_{0,D\setminus D_{el}}\big)^2 + \Big(\|\text{curl}\, \mathbf{u}\|^2_{0,D_{ml}} + \|\mathbf{u}\|^2_{0,D_{el}} + \|\mathbf{n} \times \mathbf{u} \times \mathbf{n}\|^2_{0,\Gamma}\Big)^2 \right).
\end{aligned}$$

But for any $s, t \in \mathbb{R}$ we have that $s^2 + t^2 \geq (1/2)(s+t)^2$. Thus $|a(\mathbf{u},\mathbf{u})|^2 \geq \frac{C_{10}}{2} \|\mathbf{u}\|^4_{V,\Omega}$.

10 Appendix B

In the statement of Theorem 3 a particular role was played by condition H9. This was not by chance. As a matter of fact, conditions H1-H6 were introduced to define Problem 1 and H7 simply requires that all materials involved are passive. Thus H9 and H10 are the only conditions restricting the set of models which can be dealt with our theory. However, as already pointed out, weakening condition H10 (which would be possible also in the framework of the present work) is not worthy because it does not widen significantly the covering of practical applications. Thus the only hypotheses preventing the application of our results to a bigger number of models of practical interest is H9. From this point of view, the reader could wonder whether our results of Section 4 (and then of Sections 5 and 6) can be generalized to cover, with the same theory, other cases. In this appendix we provide some indications on the fact that our results are in a certain sense as sharp as possible and that to obtain more general statements other approaches should be considered.

In order to define in which sense we will be able to provide sharper results, let us firstly point out that condition H9 can be violated even if just some of the quadratic forms involved in the definitions of J_{ml}, J_{mp}, J_{mn}, J_{el}, J_{ep}, J_{en} vanish somewhere. For instance, in order that this happens it is sufficient that $\frac{\mu^*(\mathbf{x})-\mu(\mathbf{x})}{2j} = 0$ at $\mathbf{x} = \mathbf{x}_1 \in \Omega_i$, $\mathbf{x} = \mathbf{x}_2 \in \Omega_j$, $i, j \in M$, $i \neq j$ and $i \in J_{mp}$, $j \in J_{mn}$. Cases like the indicated one prevent any further considerations and will not be considered in the following. Furtunately, they do not represent situations of great physical interest.

In order to exclude them and deduce a subset of physical models allowing sharper statements than those in Section 4 (and then in Sections 5 and 6), let us suppose that in the subdomains Ω_i, $i \in M \setminus J_{ml}$ (respectively, $i \in M \setminus J_{el}$) where "the possible losses hidden" in μ (respectively, ε) are not "uniformly positive" no losses at all are actually modeled in μ (respectively, ε). In a more precise form, we assume

H19. $\mathbf{v}^* \frac{\mu^*(\mathbf{x})-\mu(\mathbf{x})}{2j}\mathbf{v} = 0$, $\forall \mathbf{v} \in \mathbb{C}^3$, $\forall \mathbf{x} \in \bigcup_{i \in M\setminus J_{ml}} \Omega_i$,

H20. $\mathbf{v}^* \frac{\varepsilon^*(\mathbf{x})-\varepsilon(\mathbf{x})}{2j}\mathbf{v} = 0$ $\forall \mathbf{v} \in \mathbb{C}^3$, $\forall \mathbf{x} \in \bigcup_{i \in M\setminus J_{el}} \Omega_i$.

Under these additional assumptions excluding models like the ones indicated above it is possible to obtain interesting results when one of the following conditions is satisfied:

H21. $J_{mp} \neq \emptyset$ and $J_{mn} \neq \emptyset$

H22. $J_{ep} \neq \emptyset$ and $J_{en} \neq \emptyset$

H23. $J_{mp} \neq \emptyset$ and $J_{ep} \neq \emptyset$

H24. $J_{mn} \neq \emptyset$ and $J_{en} \neq \emptyset$.

In particular we have:

Theorem 8. *Whenever H1, H2, H3, H4, H6 and H8 are satisfied, the sequilinear form $a(\mathbf{u}, \mathbf{v})$ is not coercive on V if conditions H19 and H21 hold true.*

Proof. Hypotheses H1, H2, H3, H4, H6 and H8 are used to give a meaning to the sesquilinear form $a(\mathbf{u}, \mathbf{v})$, as shown in the proof of Theorem 2. In order to prove that such sesquilinear form is not coercive under the hypotheses indicated in the statement of the theorem, by using H21 we can assume, without loss of generality, that $\frac{\mu+\mu^*}{2}$ is uniformly positive definite on Ω_1 and that $\frac{\mu+\mu^*}{2}$ is uniformly negative definite on Ω_2. By the very definition of J_{mp} and J_{mn} and by H19 we deduce that $\frac{\mu^*-\mu}{2j}$ is the zero matrix-valued function on $\Omega_1 \cup \Omega_2$. Let us consider Ω_1 and Ω_2 as empty cavities with a perfectly conducting boundary and define two sequences of eigenpairs $\{(\omega_{1n}, \mathbf{u}_{1n})\}$, $\{(\omega_{2n}, \mathbf{u}_{2n})\}$ with increasing eigenvalues which therefore satisfy

$$(\operatorname{curl} \mathbf{u}_{1n}, \operatorname{curl} \mathbf{u}_{1n})_{0,\Omega_1} = \omega_{1n}^2 \varepsilon_0 \mu_0 (\mathbf{u}_{1n}, \mathbf{u}_{1n})_{0,\Omega_1}, \ \forall n \in \mathbb{N} \tag{22}$$

$$(\operatorname{curl} \mathbf{u}_{2n}, \operatorname{curl} \mathbf{u}_{2n})_{0,\Omega_2} = \omega_{2n}^2 \varepsilon_0 \mu_0 (\mathbf{u}_{2n}, \mathbf{u}_{2n})_{0,\Omega_2}, \ \forall n \in \mathbb{N}, \tag{23}$$

$$\lim_{n\to\infty} \omega_{1n} = +\infty \tag{24}$$

and

$$\lim_{n\to\infty} \omega_{2n} = +\infty. \tag{25}$$

Since all eigenfunctions are defined up to an arbitrary complex constant factor we can choose $\mathbf{u}_{1n}$ such that $\|\text{curl}\,\mathbf{u}_{1n}\|_{0,\Omega_1} = 1\ \forall n \in \mathbb{N}$, so that by conditions (22) and (24)

$$\lim_{n\to\infty} \|\mathbf{u}_{1n}\|^2_{0,\Omega_1} = 0. \tag{26}$$

Let $\zeta_3 = \frac{\mu^{-1}+(\mu^{-1})^*}{2}$. By H4, H8 and Theorem 1, ζ_3 has a meaning and is bounded on Ω_i, $i \in M$. In particular, since $\frac{\mu+\mu^*}{2}$ is uniformly positive definite on Ω_1 and $\frac{\mu+\mu^*}{2}$ is uniformly negative definite on Ω_2 we deduce by Theorem 1 that $\zeta_3|_{\Omega_1}$ is uniformly positive definite and $\zeta_3|_{\Omega_2}$ is uniformly negative definite. Then we define the amplitude of $\mathbf{u}_{2n}$ by

$$-(\zeta_3|_{\Omega_2}\text{curl}\,\mathbf{u}_{2n}, \text{curl}\,\mathbf{u}_{2n})_{0,\Omega_2} = (\zeta_3|_{\Omega_1}\text{curl}\,\mathbf{u}_{1n}, \text{curl}\,\mathbf{u}_{1n})_{0,\Omega_1} \tag{27}$$

and we deduce that $\exists\exists C_1, C_2 > 0$ such that

$$\begin{aligned} C_1\|\text{curl}\,\mathbf{u}_{2n}\|^2_{0,\Omega_2} \le -(\zeta_3|_{\Omega_2}\text{curl}\,\mathbf{u}_{2n}, \text{curl}\,\mathbf{u}_{2n})_{0,\Omega_2} = \\ (\zeta_3|_{\Omega_1}\text{curl}\,\mathbf{u}_{1n}, \text{curl}\,\mathbf{u}_{1n})_{0,\Omega_1} \le C_2\|\text{curl}\,\mathbf{u}_{1n}\|^2_{0,\Omega_1} = C_2, \end{aligned} \tag{28}$$

so that, by using also (23) and (25),

$$\lim_{n\to\infty} \|\mathbf{u}_{2n}\|^2_{0,\Omega_2} = 0. \tag{29}$$

We define a sequence $\{\mathbf{v}_n\}$ as $\mathbf{v}_n = \mathbf{0}$ on all subregion $\Omega_i, i = 3, \dots, m$, $\mathbf{v}_n = \mathbf{u}_{1n}$ on Ω_1 and $\mathbf{v}_n = \mathbf{u}_{2n}$ on Ω_2. Note that $\mathbf{v}_n \in V$ since its restrictions to Ω_i belongs to $H(\text{curl}, \Omega_i)$, for all $i \in M$, it is tangentially continuous and has trivial tangential components on the boundary.

We have $\|\mathbf{v}_n\|^2_{V,\Omega} > \|\text{curl}\,\mathbf{u}_{1n}\|^2_{0,\Omega_1} = 1\ \forall n \in \mathbb{N}$. Moreover,

$$\begin{aligned} &|a(\mathbf{v}_n, \mathbf{v}_n)|^2 = \\ &\quad |(\mu^{-1}\text{curl}\,\mathbf{v}_n, \text{curl}\,\mathbf{v}_n)_{0,\Omega} - \omega^2(\varepsilon\mathbf{v}_n, \mathbf{v}_n)_{0,\Omega} + j\omega(\xi\, \mathbf{n}\times\mathbf{v}_n\times\mathbf{n}, \mathbf{n}\times\mathbf{v}_n\times\mathbf{n})_{0,\Gamma}|^2 = \\ &\quad |(\mu^{-1}\text{curl}\,\mathbf{v}_n, \text{curl}\,\mathbf{v}_n)_{0,\Omega_1\cup\Omega_2} - \omega^2(\varepsilon\mathbf{v}_n, \mathbf{v}_n)_{0,\Omega_1\cup\Omega_2}|^2 \end{aligned} \tag{30}$$

since the tangential components are trivial and $\mathbf{v}_n = \mathbf{0}$ on all subregion $\Omega_i, i = 3, \dots, m$.

Let $\zeta_4 = \frac{(\mu^{-1})^*-\mu^{-1}}{2j}$. Since, as already pointed out, $\frac{\mu^*-\mu}{2j}$ is the zero matrix-valued function on $\Omega_1 \cup \Omega_2$, by Theorem 1 we obtain $\zeta_4 = 0$ on $\Omega_1 \cup \Omega_2$. Thus we deduce

$$\begin{aligned} |a(\mathbf{v}_n, \mathbf{v}_n)|^2 = \left|(\zeta_3\text{curl}\,\mathbf{v}_n, \text{curl}\,\mathbf{v}_n)_{0,\Omega_1\cup\Omega_2} - \omega^2(\varepsilon\mathbf{v}_n, \mathbf{v}_n)_{0,\Omega_1\cup\Omega_2}\right|^2 = \\ \left|(\zeta_3|_{\Omega_1}\text{curl}\,\mathbf{u}_{1n}, \text{curl}\,\mathbf{u}_{1n})_{0,\Omega_1} + (\zeta_3|_{\Omega_2}\text{curl}\,\mathbf{u}_{2n}, \text{curl}\,\mathbf{u}_{2n})_{0,\Omega_2} - \omega^2(\varepsilon\mathbf{v}_n, \mathbf{v}_n)_{0,\Omega_1\cup\Omega_2}\right|^2 \end{aligned} \tag{31}$$

so that by equation (27)

$$|a(\mathbf{v}_n, \mathbf{v}_n)|^2 = \omega^4|(\varepsilon\mathbf{v}_n, \mathbf{v}_n)_{0,\Omega_1\cup\Omega_2}|^2. \tag{32}$$

Then by H3, (26) and (29) we deduce $\lim_{n\to\infty}|a(\mathbf{v}_n, \mathbf{v}_n)|^2 = 0$ and since $\|\mathbf{v}_n\|^2_{V,\Omega} > 1\ \forall n \in \mathbb{N}$ the sesquilinear form cannot be coercive on V. □

Theorem 9. *Whenever H1, H2, H3, H4, H6 and H8 are satisfied, the sequilinear form $a(\mathbf{u}, \mathbf{v})$ is not coercive on V if conditions H20 and H22 hold true.*

Proof. Hypotheses H1, H2, H3, H4, H6 and H8 are used to give a meaning to the sesquilinear form $a(\mathbf{u}, \mathbf{v})$, as shown in the proof of Theorem 2. In order to prove that such sesquilinear form is not coercive under the hypotheses indicated in the statement of the theorem, by using H22 we can assume, without loss of generality, that $\zeta_5 = \frac{\varepsilon+\varepsilon^*}{2}$ is uniformly positive definite on Ω_1 and that ζ_5 is uniformly negative definite on Ω_2. By the very definition of J_{ep} and J_{en} and by H20 we deduce that $\zeta_6 = \frac{\varepsilon^*-\varepsilon}{2j}$ is the zero matrix-valued function on $\Omega_1 \cup \Omega_2$. Let us consider $\varphi_1 \in H^1_0(\Omega_1)$ and

$\psi \in H_0^1(\Omega_2)$ such that $\|\mathrm{grad}\,\varphi_1\|_{0,\Omega_1} = 1$ and $\|\mathrm{grad}\,\psi\|_{0,\Omega_2} = 1$. Finally, define $\varphi_2 = C\psi$, $C \in \mathbb{R}$ so that

$$-(\zeta_5\,\mathrm{grad}\,\varphi_2, \mathrm{grad}\,\varphi_2)_{0,\Omega_2} = (\zeta_5\,\mathrm{grad}\,\varphi_1, \mathrm{grad}\,\varphi_1)_{0,\Omega_1} \tag{33}$$

and $\mathbf{v}$ as $\mathbf{v} = \mathbf{0}$ on all subregion $\Omega_i, i = 3, \ldots, m$, $\mathbf{v} = \mathrm{grad}\,\varphi_1$ on Ω_1 and $\mathbf{v} = \mathrm{grad}\,\varphi_2$ on Ω_2. Note that $\mathbf{v} \in V$ since its restrictions to Ω_i belongs to $H(\mathrm{curl}, \Omega_i)$, for all $i \in M$, it is tangentially continuous and has trivial tangential components on the boundary.

On the one hand we have that $\|\mathbf{v}\|^2_{V,\Omega} > \|\mathrm{grad}\,\varphi_1\|^2_{0,\Omega_1} = 1$. On the other hand,

$$\begin{aligned}|a(\mathbf{v},\mathbf{v})|^2 = &\\ &|(\mu^{-1}\mathrm{curl}\,\mathbf{v}, \mathrm{curl}\,\mathbf{v})_{0,\Omega} - \omega^2(\varepsilon\mathbf{v},\mathbf{v})_{0,\Omega} + j\omega(\xi\,\mathbf{n}\times\mathbf{v}\times\mathbf{n}, \mathbf{n}\times\mathbf{v}\times\mathbf{n})_{0,\Gamma}|^2 = |\omega^2(\varepsilon\mathbf{v},\mathbf{v})_{0,\Omega_1\cup\Omega_2}|^2.\end{aligned} \tag{34}$$

since $\mathrm{curl}\,\mathbf{v} = \mathbf{0}$ in Ω, the tangential components are trivial on Γ and $\mathbf{v} = \mathbf{0}$ on all subregion $\Omega_i, i = 3, \ldots, m$. By using $\varepsilon = \zeta_5 - j\zeta_6$, their properties and equations (33) and (34) we deduce

$$|a(\mathbf{v},\mathbf{v})|^2 = \omega^4|(\zeta_5\,\mathrm{grad}\,\varphi_1, \mathrm{grad}\,\varphi_1)_{0,\Omega_1} + (\zeta_5\,\mathrm{grad}\,\varphi_2, \mathrm{grad}\,\varphi_2)_{0,\Omega_2}|^2 = 0. \tag{35}$$

□

Theorem 10. *Whenever H1, H2, H3, H4, H6 and H8 are satisfied, the sequilinear form $a(\mathbf{u},\mathbf{v})$ is not coercive on V if either H19, H20 and H23 or H19, H20 and H24 hold true.*

Proof. Hypotheses H1, H2, H3, H4, H6 and H8 are used to give a meaning to the sesquilinear form $a(\mathbf{u},\mathbf{v})$, as shown in the proof of Theorem 2. In order to prove that such sesquilinear form is not coercive under the hypotheses indicated in the statement of the theorem, let us point out that when H23 holds true we can assume, without loss of generality, that $i = 1 \in J_{mp}$ and $i = 2 \in J_{ep}$. Analogously, when H24 is satisfied we can assume that $i = 1 \in J_{mn}$ and $i = 2 \in J_{en}$. We define $\zeta_5 = \frac{\varepsilon+\varepsilon^*}{2}$ when H23 holds true and $\zeta_5 = -\frac{\varepsilon+\varepsilon^*}{2}$ when H24 is satisfied. By using H4, H23 (or H24) and Theorem 1 we deduce that μ^{-1} has a meaning on Ω_1 and that its entries are bounded on Ω_1. Thus, we can define $\zeta_3 = \frac{\mu^{-1}+(\mu^{-1})^*}{2}$ when H23 holds true and $\zeta_3 = -\frac{\mu^{-1}+(\mu^{-1})^*}{2}$ when H24 is satisfied. It is important to note that in both cases ζ_3 is uniformly positive definite on Ω_1 and ζ_5 is uniformly positive definite on Ω_2.

Let us now consider, on the one hand, $\varphi \in H_0^1(\Omega_2)$ such that $\|\mathrm{grad}\,\varphi\|_{0,\Omega_2} = C_1 > 0$ and, on the other hand, a sequence of eigenpairs $(\omega_n, \mathbf{u}_n)$ with increasing eigenvalues satisfying $\mathbf{u}_n \in H_0(\mathrm{curl}, \Omega_1)$,

$$(\zeta_3\,\mathrm{curl}\,\mathbf{u}_n, \mathrm{curl}\,\mathbf{u}_n)_{0,\Omega_1} = \omega_n^2\,\varepsilon_0\,(\mathbf{u}_n, \mathbf{u}_n)_{0,\Omega_1} \tag{36}$$

and

$$\lim_{n\to\infty} \omega_n = +\infty. \tag{37}$$

Since $\mathbf{u}_n$ is determined up to an arbitrary complex constant factor we can choose $\mathbf{u}_n$ in such a way that

$$(\zeta_3\,\mathrm{curl}\,\mathbf{u}_n, \mathrm{curl}\,\mathbf{u}_n)_{0,\Omega_1} = \omega^2(\zeta_5\,\mathrm{grad}\,\varphi, \mathrm{grad}\,\varphi)_{0,\Omega_2}. \tag{38}$$

By using this normalization, the fact that ζ_3 has bounded entries on Ω_1, equation (38) and the uniformly positive definiteness of ζ_5 on Ω_2, we obtain that $\exists\exists C_2, C_3 > 0$:

$$C_3\|\mathrm{curl}\,\mathbf{u}_n\|^2_{0,\Omega_1} \geq (\zeta_3\,\mathrm{curl}\,\mathbf{u}_n, \mathrm{curl}\,\mathbf{u}_n)_{0,\Omega_1} = \omega^2(\zeta_5\,\mathrm{grad}\,\varphi, \mathrm{grad}\,\varphi)_{0,\Omega_2} \geq \omega^2 C_2\|\mathrm{grad}\,\varphi\|^2_{0,\Omega_2} \tag{39}$$

and, moreover, by using (36), (38), H3 and (37)

$$\begin{aligned}\lim_{n\to\infty}(\mathbf{u}_n, \mathbf{u}_n)_{0,\Omega_1} = \lim_{n\to\infty}\frac{(\zeta_3\,\mathrm{curl}\,\mathbf{u}_n, \mathrm{curl}\,\mathbf{u}_n)_{0,\Omega_1}}{\omega_n^2\,\varepsilon_0} = &\\ \lim_{n\to\infty}\frac{\omega^2(\zeta_5\,\mathrm{grad}\,\varphi, \mathrm{grad}\,\varphi)_{0,\Omega_2}}{\omega_n^2\,\varepsilon_0} = \frac{\omega^2}{\varepsilon_0}(\zeta_5\,\mathrm{grad}\,\varphi, \mathrm{grad}\,\varphi)_{0,\Omega_2}\lim_{n\to\infty}\frac{1}{\omega_n^2} = 0&\end{aligned} \tag{40}$$

which in turn implies, together with H3

$$\lim_{n\to\infty} (\varepsilon\, \mathbf{u}_n, \mathbf{u}_n)_{0,\Omega_1} = 0. \tag{41}$$

We now define a sequence $\{\mathbf{v}_n\}$, $\mathbf{v}_n \in H_0(\text{curl}, \Omega) \subset V$ $\forall n \in \mathbb{N}$, as follows

$$\mathbf{v}_n = \begin{cases} \text{grad}\,\varphi & \text{in } \Omega_2 \\ \mathbf{u}_n & \text{in } \Omega_1 \\ 0 & \text{in } \Omega \setminus (\Omega_2 \cup \Omega_1)\,. \end{cases} \tag{42}$$

By using the fact that $\mathbf{n} \times \mathbf{v}_n = \mathbf{0}$ on Γ, equation (39) and definition (42) we deduce that

$$\begin{aligned} \|\mathbf{v}_n\|^2_{V,\Omega} &= \|\text{curl}\,\mathbf{v}_n\|^2_{0,\Omega} + \|\mathbf{v}_n\|^2_{0,\Omega} + \|\mathbf{n} \times \mathbf{v}_n\|^2_{0,\Gamma} = \|\text{curl}\,\mathbf{v}_n\|^2_{0,\Omega} + \|\mathbf{v}_n\|^2_{0,\Omega} = \\ &\|\text{curl}\,\mathbf{v}_n\|^2_{0,\Omega_2} + \|\text{curl}\,\mathbf{v}_n\|^2_{0,\Omega_1} + \|\mathbf{v}_n\|^2_{0,\Omega_2} + \|\mathbf{v}_n\|^2_{0,\Omega_1} = \\ &\|\text{curl}\,\text{grad}\,\varphi\|^2_{0,\Omega_2} + \|\text{curl}\,\mathbf{u}_n\|^2_{0,\Omega_1} + \|\text{grad}\,\varphi\|^2_{0,\Omega_2} + \|\mathbf{u}_n\|^2_{0,\Omega_1} = \\ &\|\text{curl}\,\mathbf{u}_n\|^2_{0,\Omega_1} + \|\text{grad}\,\varphi\|^2_{0,\Omega_2} + \|\mathbf{u}_n\|^2_{0,\Omega_1} \geq \\ &\|\text{curl}\,\mathbf{u}_n\|^2_{0,\Omega_1} + \|\text{grad}\,\varphi\|^2_{0,\Omega_2} \geq (\omega^2 \frac{C_2}{C_3} + 1)\|\text{grad}\,\varphi\|^2_{0,\Omega_2} \geq (\omega^2 \frac{C_2}{C_3} + 1) C_1^2 > 0 \;\forall n \in \mathbb{N}. \end{aligned} \tag{43}$$

Moreover, $\mathbf{n} \times \mathbf{v}_n = \mathbf{0}$ on Γ and definition (42) imply

$$\begin{aligned} &|a(\mathbf{v}_n, \mathbf{v}_n)| = \\ &\quad \left|(\mu^{-1}\text{curl}\,\mathbf{v}_n, \text{curl}\,\mathbf{v}_n)_{0,\Omega} - \omega^2(\varepsilon\mathbf{v}_n, \mathbf{v}_n)_{0,\Omega} + j\omega(\xi\, \mathbf{n} \times \mathbf{v}_n \times \mathbf{n}, \mathbf{n} \times \mathbf{v}_n \times \mathbf{n})_{0,\Gamma}\right| = \\ &\quad \left|(\mu^{-1}\text{curl}\,\mathbf{v}_n, \text{curl}\,\mathbf{v}_n)_{0,\Omega} - \omega^2(\varepsilon\mathbf{v}_n, \mathbf{v}_n)_{0,\Omega}\right| = \\ &\quad \left|(\mu^{-1}\text{curl}\,\mathbf{v}_n, \text{curl}\,\mathbf{v}_n)_{0,\Omega_1\cup\Omega_2} - \omega^2(\varepsilon\mathbf{v}_n, \mathbf{v}_n)_{0,\Omega_1\cup\Omega_2}\right| = \\ &\quad \big|(\mu^{-1}\text{curl}\,\mathbf{u}_n, \text{curl}\,\mathbf{u}_n)_{0,\Omega_1} + (\mu^{-1}\text{curl}\,\text{grad}\,\varphi, \text{curl}\,\text{grad}\,\varphi)_{0,\Omega_2} \\ &\qquad -\omega^2(\varepsilon\mathbf{u}_n, \mathbf{u}_n)_{0,\Omega_1} - \omega^2(\varepsilon\text{grad}\,\varphi, \text{grad}\,\varphi)_{0,\Omega_2}\big| = \\ &\quad \left|(\mu^{-1}\text{curl}\,\mathbf{u}_n, \text{curl}\,\mathbf{u}_n)_{0,\Omega_1} - \omega^2(\varepsilon\mathbf{u}_n, \mathbf{u}_n)_{0,\Omega_1} - \omega^2(\varepsilon\,\text{grad}\,\varphi, \text{grad}\,\varphi)_{0,\Omega_2}\right| \leq \\ &\quad \left|(\mu^{-1}\text{curl}\,\mathbf{u}_n, \text{curl}\,\mathbf{u}_n)_{0,\Omega_1} - \omega^2(\varepsilon\,\text{grad}\,\varphi, \text{grad}\,\varphi)_{0,\Omega_2}\right| + \omega^2\,|(\varepsilon\mathbf{u}_n, \mathbf{u}_n)_{0,\Omega_1}|\,. \end{aligned} \tag{44}$$

When H23, H19 and H20 are satisfied we have $\mu^{-1} = \zeta_3$ on Ω_1 and $\varepsilon = \zeta_5$ on Ω_2 while we have $\mu^{-1} = -\zeta_3$ on Ω_1 and $\varepsilon = -\zeta_5$ on Ω_2 when H24, H19 and H20 hold true. Then in both cases

$$|a(\mathbf{v}_n, \mathbf{v}_n)| \leq \left|(\zeta_3\,\text{curl}\,\mathbf{u}_n, \text{curl}\,\mathbf{u}_n)_{0,\Omega_1} - \omega^2(\zeta_5\,\text{grad}\,\varphi, \text{grad}\,\varphi)_{0,\Omega_2}\right| + \omega^2\,|(\varepsilon\mathbf{u}_n, \mathbf{u}_n)_{0,\Omega_1}|\,. \tag{45}$$

By using equations (38) and (41) we obtain $\lim_{n\to\infty} |a(\mathbf{v}_n, \mathbf{v}_n)| = 0$, which together with (43) complete the proof. ☐

By using these intermediate results and by defining

H25. $J_{el} \cup J_{ep} \cup J_{en} = M$.

we can now state the following theorem, which is the main result of this appendix.

Theorem 11. *Whenever H1, H2, H3, H4, H5, H6, H7, H8, H25, H10, H19 and H20 are satisfied, the sequilinear form $a(\mathbf{u}, \mathbf{v})$ is coercive on V if and only if condition H9 holds true.*

Proof. The logical "or" of conditions H21, H22, H23 and H24 is

$$\begin{aligned} &\big(J_{mp} \neq \emptyset \text{ and } J_{mn} \neq \emptyset\big) \text{ or } \big(J_{ep} \neq \emptyset \text{ and } J_{en} \neq \emptyset\big) \text{ or} \\ &\qquad \big(J_{mp} \neq \emptyset \text{ and } J_{ep} \neq \emptyset\big) \text{ or } \big(J_{mn} \neq \emptyset \text{ and } J_{en} \neq \emptyset\big) \end{aligned} \tag{46}$$

which is equivalent to

$$\big(J_{mn} \neq \emptyset \text{ or } J_{ep} \neq \emptyset\big) \text{ and } \big(J_{mp} \neq \emptyset \text{ or } J_{en} \neq \emptyset\big). \tag{47}$$

Under assumption H8 the condition $J_{mn} \neq \emptyset$ is equivalent to $J_{ml} \cup J_{mp} \neq M$. But the latter is equivalent to $J_{mp} \neq M \setminus J_{ml}$ or even to $\overline{(J_{mp} = M \setminus J_{ml})}$, having denoted the logical "not" by the overline. Analogously, under assumption H8 the condition $J_{mp} \neq \emptyset$ is equivalent to $\overline{(J_{mn} = M \setminus J_{ml})}$ and, under assumption H25 condition $J_{en} \neq \emptyset$ is equivalent to $\overline{(J_{ep} = M \setminus J_{el})}$ and condition $J_{ep} \neq \emptyset$ is equivalent to $\overline{(J_{en} = M \setminus J_{el})}$.

Therefore, under assumptions H8 and H25, the logical "or" of conditions H21, H22, H23 and H24 is logically equivalent to

$$\begin{aligned}
&\left(\overline{(J_{mp} = M \setminus J_{ml})} \text{ or } \overline{(J_{en} = M \setminus J_{el})}\right) \text{ and} \\
&\qquad \left(\overline{(J_{mn} = M \setminus J_{ml})} \text{ or } \overline{(J_{ep} = M \setminus J_{el})}\right) = \\
&\qquad \overline{\left((J_{mp} = M \setminus J_{ml}) \text{ and } (J_{en} = M \setminus J_{el})\right)} \text{ and} \\
&\qquad \overline{\left((J_{mn} = M \setminus J_{ml}) \text{ and } (J_{ep} = M \setminus J_{el})\right)} = \\
&\qquad \overline{\left(\left((J_{mp} = M \setminus J_{ml}) \text{ and } (J_{en} = M \setminus J_{el})\right) \text{ or } \left((J_{mn} = M \setminus J_{ml}) \text{ and } (J_{ep} = M \setminus J_{el})\right)\right)}.
\end{aligned} \tag{48}$$

The right-hand side member is the logical negation of condition H9, as the reader can easily verify.

Thus, if conditions H8 and H25 are satisfied and condition H9 is violated we deduce that the logical or of conditions H21, H22, H23 and H24 is satisfied. This implies that when conditions H1, H2, H3, H4, H6, H8, H25, H19 and H20 are satisfied and condition H9 is violated the hypotheses of at least one of Theorems 8, 9 or 10 are satisfied and the sesquilinear form $a(\mathbf{u}, \mathbf{v})$ is not coercive in V. Therefore, when conditions H1, H2, H3, H4, H6, H8, H25, H19 and H20 are satisfied and the sesquilinear form $a(\mathbf{u}, \mathbf{v})$ is coercive in V condition H9 is necessarily satisfied.

The other implication is provided by Theorem 3. □

Thus, under the additional hypotheses H8, H25, H19 and H20, in the statement of Theorem 3 (and in the following statements) the word "if" can be replaced by "if and only if". This means that under the conditions introduced to define Problem 1 (H1-H6), when all materials involved are passive (H7) and boundary conditions of impedance gives some "leakage" of power (H10), our results cannot be generalized to cover more cases of practical interest within the class of models satisfying H8, H25, H19 and H20. In this class condition H9 results to be necessary and sufficient to identify all models which can be dealt with our theory, whereas when H8, H25, H19 or H20 are not satisfied condition H9 is just sufficient (by Theorem 3) for the same result to hold true. As already pointed out weakening condition H10 would be possible also in the framework of the present work but it is not worthy because it does not widen significantly the covering of practical applications.

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